EUROPHYSICS LETTERS

Europhys. Lett., (), pp. ()

Hall state quantization in a rotating frame

UWE R. FISCHER AND NILS SCHOPOHL

Eberhard-Karls-Universität Tübingen, Institut für Theoretische Physik
Auf der Morgenstelle 14, D-72076 Tübingen, Germany

(received ; accepted in final form)

PACS. 73.40Hm– Quantum Hall effect (integer and fractional).
PACS. 73.50−h – Electronic transport phenomena in thin films and low-dimensional structures.

Abstract. – We derive electromagnetomotive force fields for charged particles moving in a rotating Hall sample, satisfying a twofold U(1) gauge invariance principle. It is then argued that the phase coherence property of quantization of the line integral of total collective particle momentum into multiples of Planck's quantum of action is solely responsible for quantization in the Hall state. As a consequence, the height of the Hall quantization steps should remain invariant in a rapidly rotating Hall probe. Quantum Hall particle conductivities do not depend on charge and mass of the electron, and are quantized in units of the inverse of Planck's action quantum.

Modern molecular beam epitaxy enables the preparation of modulation-doped semiconductor heterostructures in which, at low enough temperatures, a high mobility two-dimensional electron gas is formed. This system is characterized by a long Thouless dephasing length l_ϕ, the distance within which phase coherence of mobile electrons is maintained. In good samples, and at low enough temperatures, the length l_ϕ reaches several micrometers, exceeding the magnetic length $l_B = \sqrt{\hbar/eB}$ for applied magnetic fields of order one Tesla. Under these conditions, it should be possible to detect noninertial effects due to rotation or acceleration of the sample as a result of the change of quantum interference conditions, since the gauge potentials of the electromagnetic and noninertial fields experienced by the electrons both appear in their collective phase. In what follows, we shall argue that quantum coherence under the influence of noninertial force fields is directly observable in the quantum Hall effect [1]. The quantum of Hall conductivity for particle transport is given by the inverse of Planck's quantum of action alone, and involves no properties specific to the electron. The arguments used to prove this result crucially rely on the existence of a collective particle momentum expressing quantum coherence. By including gauge fields other than the electromagnetic one, we promote the idea that Hall quantization is of necessity derivable from a topological quantum number related to this coherence. It will be shown that this prediction about the nature of the quantum Hall effect is verifiable within current technological means.

The main quantity of interest to us is the total particle momentum

$$\boldsymbol{p} = m\boldsymbol{v} + m\boldsymbol{\Omega} \times \boldsymbol{r} + q\boldsymbol{A}\,. \tag{1}$$

Here, $\boldsymbol{A}$ is the electromagnetic vector potential, q the charge of the electron *in vacuo*, $\boldsymbol{v}$ the particle velocity, and m the inertial mass. The body is rigidly rotating with respect to the laboratory frame at a (time dependent) angular velocity $\boldsymbol{\Omega}$. The noninertial force on a massive test particle inside a rotating Hall probe, as measured in the rotating sample frame, is then given by the standard expression

$$\boldsymbol{F} = -2m\boldsymbol{\Omega} \times \boldsymbol{v} - m\boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \boldsymbol{r} - m\partial_t \boldsymbol{\Omega} \times \boldsymbol{r}. \tag{2}$$

The first term on the right hand side represents (minus) the Coriolis force, the second one the centripetal force, and the last term is due to temporal changes of the angular velocity. The presence of this last term will prove to be crucial for our argument on proper Hall quantization in a rotating frame presented below. We omit possible additional terms on the right hand side of equation (2) due to potential forces (*e.g.* gravity) or externally imposed linear acceleration. Vector and scalar potentials associated to rotation are defined as follows

$$\boldsymbol{a} = \boldsymbol{\Omega} \times \boldsymbol{r}\,, \qquad a_0 = \frac{1}{2}\Omega^2 \boldsymbol{r}_\perp^2\,, \tag{3}$$

where $\boldsymbol{r}_\perp$ is the distance vector perpendicular to the axis of rotation. For a charged massive particle like the electron, we merge these potentials and the electromagnetic potentials into a generalized vector potential, incorporating the coupling constants charge q and mass m,

$$\mathcal{A} = q\boldsymbol{A} + m\boldsymbol{a}\,, \tag{4}$$

and a generalized scalar potential

$$\chi = -qA_0 - ma_0\,. \tag{5}$$

The sum of the generalized *electromotive* and *magnetomotive* forces, acting on an electron [2], consisting of noninertial plus proper Lorentz and electric forces, then takes on the form

$$\boldsymbol{F}_{\mathcal{L}} = \mathcal{E} + \boldsymbol{v} \times \mathcal{B}\,, \tag{6}$$

where the generalized electric and magnetic fields are

$$\begin{aligned} \mathcal{E} &= -\nabla\chi - \partial_t \mathcal{A}\,, \\ \mathcal{B} &= \nabla \times \mathcal{A}\,. \end{aligned} \tag{7}$$

As a consequence of this relation for the total force, the usual expression for the drift velocity of the charge carriers, resulting from zero total force in perpendicular electric and magnetic fields, experiences the obvious modification that $\boldsymbol{E} \to \mathcal{E}$ and $\boldsymbol{B} \to \mathcal{B}$, so that $\boldsymbol{v}_D = \mathcal{E} \times \mathcal{B}/\mathcal{B}^2$.

The force fields displayed in equations (6) and (7) give a theory possessing in effect two U(1) gauge symmetries. The standard U(1) from electromagnetism, with coupling constant q (charge), and another U(1) gauge symmetry, with coupling constant m (inertial rest mass). The gauge potential of this second U(1) has a scalar part a_0 and a vectorial part $\boldsymbol{a}$. The homogeneous Maxwell equations $\mathrm{rot}\,\mathcal{E} = -\partial_t \mathcal{B}$ and $\mathrm{div}\,\mathcal{B} = 0$ then follow from the existence of the potentials $\mathcal{A}$ and χ in (4) and (5). That the Faraday law holds is due to our admitting a variation of the angular velocity with time and the resulting last force term in (2). This important term, leading to gauge invariance in explicitly time dependent situations, has not been considered in [3], where the Hall effect under rotation, without the simultaneous existence of a magnetic field, was investigated.

The gauge invariant particle current induced by the electromotive force field $\mathcal{E}$ is in linear response ($a, b \in \{x, y\}$):

$$\begin{aligned} \boldsymbol{J}_a^{\mathrm{ind}} &= \vec{\boldsymbol{\sigma}}_{ab}\mathcal{E}_b \\ &= \vec{\boldsymbol{\sigma}}_{ab}\left(q\boldsymbol{E} + m\boldsymbol{g}\right)_b\,, \end{aligned} \tag{8}$$

where $\boldsymbol{g} = \nabla a_0 - \partial_t \boldsymbol{a}$ represents the 'electric' part of the mechanical acceleration experienced by the electron. In the present case this acceleration is purely caused by rotation, and takes the form $\boldsymbol{g} = -\boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \boldsymbol{r} - \partial_t \boldsymbol{\Omega} \times \boldsymbol{r}$.

Observe that the left hand side of equation (8) contains the number current density rather than the electric current density and that, dimensionally, the particle conductivity $[\tilde{\sigma}] = [\sigma_{el}/q^2]$. In the case of two coupling constants, m and q, it is the number of particles crossing (in two spatial dimensions) a line of unit length per unit time, which is the relevant observable. This quantity is proportional to the electromotive force field $\mathcal{E}$, which causes these particles to move. If we were to use the transport coefficient $\sigma_{\rm el}$ and the transport equation $\boldsymbol{J}_{\rm el}^{\rm ind} = \boldsymbol{\sigma}_{\rm el}\boldsymbol{E}$, a rotating Hall sample does not yield the sharp conductivity quantization steps observed in nonrotating samples. We will show below that Hall state quantization is, according to equation (16), to be expressed in the particle Hall conductivity $\tilde{\sigma}_{xy}$ occuring in relation (8).

Evidence for the necessity of using the particle transport equation (8) comes from the existence of the London field in superconductors. Complete expulsion of the field $\mathcal{B}$ deep inside in a superconductor requires the particle conductivity $\vec{\boldsymbol{\sigma}}$ to have a contribution proportional to $1/i\omega$, which yields a term on the right hand side of (8), proportional to the generalized vector potential $\mathcal{A}$. Corresponding to complete Meissner type screening, $\mathcal{B} = {\rm rot}\,\mathcal{A} = q\boldsymbol{B} + 2m\boldsymbol{\Omega} = 0$, the London spontaneous field $\boldsymbol{B}_L$ then takes the value

$$\boldsymbol{B}_L = -2\frac{m}{q}\boldsymbol{\Omega}\,. \tag{9}$$

This relation corresponds to zero winding number of the phase θ, cf. equations (10)–(12) below. Equation (9) has been verified experimentally already 35 years ago [4], in an experiment in which it was used to infer the Compton wavelength of superconducting electrons. For the linear in velocity (nonrelativistic) limit and in a superconductor, the Cooper pair mass m equals twice the electron inertial rest mass *in vacuo*, $2m_e$ (the outcome of a more recent high precision experiment using a rotating superconducting niobium ring [5] has been $m/2m_e = 1.000084(21)$). If we insert on the left hand side of the equation (9) the bare electron values $m = 2m_e$ and $q = -2e$ ($e > 0$), we have $\boldsymbol{B}_L = (1.14 \cdot 10^{-11}{\rm Tesla} \cdot {\rm sec})\,\boldsymbol{\Omega}$. Only the ratio of m and q enters the London induced magnetic flux strength. In a quantum Hall liquid, where the "elementary" quanta are $q = -e$, $m = m_e$ and $\phi_0 = 2\pi\hbar/e$, instead of $q = -2e$, $m = 2m_e$ and $\phi_0 = 2\pi\hbar/2e$ in a superconductor, the London flux strength thus takes for a given $\boldsymbol{\Omega}$ the same value (9) like in the superconductor. That the mass is exactly the bare inertial mass to linear order in the particle velocity is independently substantiated by a recent discussion of the London equation in [6], by using (thermodynamic) arguments different from our gauge invariance argument. If the quantum Hall fluid is described in a (relativistic) theory with interactions, the particle mass is to be replaced by the chemical potential, cf. [10], but the identity of m with the bare mass to linear order in the velocity still persists.

Quantum coherence properties enter the stage if we require for the line integral of collective particle momentum along a closed path

$$\oint \boldsymbol{p} \cdot d\boldsymbol{r} = N_v 2\pi\hbar\,, \tag{10}$$

where N_v is the winding number of phase θ, such that the total canonical momentum

$$\begin{aligned} \boldsymbol{p} &\equiv \hbar\nabla\theta \\ &= m\boldsymbol{v} + m\boldsymbol{\Omega} \times \boldsymbol{r} + q\boldsymbol{A} \\ &= m\boldsymbol{v} + \mathcal{A}\,. \end{aligned} \tag{11}$$

The uniqueness condition of the collective phase represented in (10) then leads, if we take a

path in the bulk of the electron liquid, for which the integral of $m\boldsymbol{v}$ may be neglected, to the quantization of the sum of a Sagnac flux [7, 8] and the magnetic flux:

$$\begin{aligned}\Phi &= q\oint \boldsymbol{A}\cdot d\boldsymbol{r} + m\oint \boldsymbol{\Omega}\times\boldsymbol{r}\cdot d\boldsymbol{r} \\ &= \iint \mathcal{B}\cdot d\boldsymbol{S} = N_v 2\pi\hbar\,. \end{aligned} \tag{12}$$

This flux quantization rule associated with the field $\mathcal{B}$ corresponds to the fact that a *vortex* is fundamentally characterized by the winding number N_v alone [10]. No properties of the medium in which it lives, in particular the mass and charge of the medium's constituents, enter the quantum of generalized flux.

Consider now the (purely magnetic) filling factor of a nonrotating two-dimensional electronic system of areal density n_{2D} in a large magnetic field B at low temperatures. The ratio $\nu = n_{2D}/(B/\phi_0)$, where $\phi_0 = 2\pi\hbar/e$, gives the inverse of the number of singly quantized ($N_v = 1$) magnetic flux quanta available per electron. The Hall resistance is quantized into $R_H = (2\pi\hbar/e^2)\nu^{-1} = R_K/\nu$ (the von Klitzing constant $R_K = 25812.807\ \Omega$), with integer or fractional ν [9]. We have seen that the dynamical, generalized magnetic force field occuring in the Hamiltonian in the noninertial rotating state is $\mathcal{B}$. The solution of the Landau problem for the electronic energy levels in a magnetic field thus refers to a Hamiltonian $H = (\boldsymbol{p}-\mathcal{A})^2/2m + \chi$, depending on the generalized vector potential $\mathcal{A}$ and magnetic field $\mathcal{B}$. Hence, in the noninertial case, the Landau level degeneracy per unit area is $\mathcal{B}/(2\pi\hbar)$, and contains the generalized magnetomotive field $\mathcal{B}$ instead of the magnetic flux strength $\boldsymbol{B}$. By assigning this value to the degeneracy, use is made of the fact that the "magnetic length" associated with rotation, $l_\Omega \equiv \sqrt{\hbar/(2m\Omega)}$, is much larger than the Thouless length as well as the proper magnetic length, $l_\Omega \gg l_\phi > l_B$ [11]. As a consequence, the linear dependence on position of the part of $\mathcal{E}$ associated with rotation, given by $-m\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\boldsymbol{r} - m\partial_t\boldsymbol{\Omega}\times\boldsymbol{r}$, does not lift the Landau level degeneracy beyond the broadening of the levels already taking place due to scattering. Under the condition $l_\Omega \gg l_\phi > l_B$, the filling factors assigned to the Landau levels are

$$\nu_{\mathcal{B}} = \frac{n_{2D}}{\mathcal{B}/(2\pi\hbar)}\,. \tag{13}$$

The Faraday law $\mathrm{rot}\,\mathcal{E} = -\partial_t\mathcal{B}$, telling us how the flux strength corresponding to the vector potential $\mathcal{A}$ changes in time, is in its integrated form

$$\oint_\Gamma \mathcal{E}\cdot d\boldsymbol{r} = -\frac{d\Phi}{dt}\,. \tag{14}$$

Consider the adiabatic process of slowly turning on a flux quantum $2\pi\hbar$ in a nondissipative Hall state, which has $\tilde\sigma_{xx} = \tilde\sigma_{yy} = 0 = \tilde\rho_{xx} = \tilde\rho_{yy}$, and the antisymmetry property $\tilde\sigma_{xy} = -\tilde\sigma_{yx}$ [1]. The path Γ is led around the flux tube. The induced current then obeys $\hat z\times\boldsymbol{J}^{\rm ind} = \tilde\sigma_{xy}\mathcal{E}$, and the number of particles N inside the area enclosed by Γ changes according to

$$\frac{dN}{dt} = \tilde\sigma_{xy}\frac{d\Phi}{dt}\,. \tag{15}$$

After a single quantum of generalized flux $2\pi\hbar$ has been added, and because the number of particles is integral, this implies that the off-diagonal part of the particle transport conductivity defined in (8) is quantized according to

$$\tilde\sigma_{xy} = \nu_{\mathcal{B}}/2\pi\hbar\,, \tag{16}$$

with $\nu_{\mathcal{B}}$ an integer [12]. Hence, to summarize this derivation, the generalized Faraday law in a rotating frame gives in a nondissipative Hall state the quantization of the Hall conductivity if and only if the moving generalized flux in (12) is quantized in units of the action quantum. This argument is similar to the one given by Laughlin [13], extended to a rotating frame.

The quantization of the Hall resistance into R_K/ν for nonrotating samples has been measured to an absolute accuracy of a few parts in 10^{-8} for an individual, specific sample, whereas in a comparative study of different materials, a relative accuracy of about 10^{-10} of the ratio of Hall resistances has been achieved [14]. Considering that magnetic fields in quantum Hall experiments cover a range $B \sim 1 \cdots 30$ Tesla, this implies that with a rotation rate of the Hall sample of, say, $\Omega = 10^3\,\mathrm{sec}^{-1} \cdots 10^4\,\mathrm{sec}^{-1}$, the Sagnac contribution in (12) is large enough to verify if Hall quantization experiences a change if the sample rotates. For the comparative measurement, the rotation rates required are lower by about two orders of magnitude. At the very low temperatures of order 10^{-3} K needed for superfluid ^{3}He, rotation rates of the cryostat of order $\Omega \sim 10\ \mathrm{sec}^{-1}$ already have been realized [15]. In relation to superfluid ^{3}He, it is also worthwhile to mention here that for thin ^{3}He-A films, an electrically neutral system, a half integer quantum Hall effect, owing to a topological invariant of the p-wave order parameter in this system, has been suggested [16].

We point out that with respect to the interpretation of experimental results, it should be borne in mind that the force fields in (6) and (7) refer to the rotating sample frame, and not to the laboratory frame.

The experiment proposed, then, consists in a comparison of the quantum Hall resistances in the reference frame of a rotating sample as well as in a nonrotating sample. If identical quantization results are obtained, this yields direct proof that what is actually observed in the Hall experiment is the quantization according to (16), rather than quantization into $e^2/2\pi\hbar$. The electric charge of the electron, the U(1) coupling constant of electromagnetism, enters if we count the number of particles arriving at the Hall contacts, by ascribing to their transport properties an electric conductance in an electric circuit. What is invariantly measured, though, is the induced number current in (8). Particle number currents can be distinguished from conventional electric currents as follows. Whereas electrical currents are measured by (classical) means of an impedance and the ensuing voltage drop, intrinsic particle currents can be measured if we use tunneling contacts, for which the wave-particle duality embodied in the tunneling process ensures that particles are counted.

We stress that if we were to use a Chern-Simons effective field theory for a description of the quantum Hall effect [17], such an effective description is expressible within our approach. For that purpose, one uses a term in the action density of the form

$$-\,(J^{\mathrm{ind}})^\mu \mathcal{A}_\mu + \frac{1}{4}\tilde{\sigma}_{xy}\epsilon^{\alpha\beta\gamma}\mathcal{A}_\alpha \mathcal{F}_{\beta\gamma}\,, \tag{17}$$

where $\epsilon^{\alpha\beta\gamma} = \pm 1$ (with the sign convention $\epsilon^{0xy} = +1$) is the unit antisymmetric symbol in three space-time dimensions and $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$ is the field tensor constructed from $\mathcal{E}$ and $\mathcal{B}$. Such a term reproduces the induced current (8) in the nondissipative Hall state as a result of varying the action with respect to $\mathcal{A}_i$. Integration of the zeroth component of the complete field equations $(J^{\mathrm{ind}})^\mu = \frac{1}{2}\tilde{\sigma}_{xy}\epsilon^{\mu\alpha\beta}\mathcal{F}_{\alpha\beta}$, giving the induced density

$$\rho^{\mathrm{ind}} = \tilde{\sigma}_{xy}\mathcal{B}_z\,, \tag{18}$$

tells us that each particle associated with ρ^{ind} carries generalized flux $1/\tilde{\sigma}_{xy} = 2\pi\hbar/\nu_{\mathcal{B}}$.

In conclusion, the quantization of the Hall particle conductivity under rotation has been derived by invoking the following basic arguments. (i) The generalized Lorentz force equation (6), containing the invariant field strengths $\mathcal{E}$ and $\mathcal{B}$, and describing the motion of the charge

carriers, is valid. (ii) The flux conservation law for the magnetomotive field $\mathcal{B}$ in (14) holds true. (iii) We consider a nondissipative Hall state, which has $\tilde{\sigma}_{xx} = \tilde{\sigma}_{yy} = 0 = \tilde{\rho}_{xx} = \tilde{\rho}_{yy}$. (iv) A necessary condition for Hall quantization to hold is that the total collective canonical particle momentum is derivable from a collective quantum phase, $\boldsymbol{p} \equiv \hbar\nabla\theta$, such that the Bohr-Sommerfeld type integral of this momentum is quantized into units of Planck's action quantum. The translation of the conventional quantum Hall problem into a rotating frame elucidates and strengthens the point of view that it is essentially a basic Hamiltonian quantity in phase space, the closed action integral of the collective particle momentum $\boldsymbol{p}$, which yields Hall state quantization.

An experimental proof or disproof of the quantization rule (16) in a rotating system will show if our assertion about the nature of the quantum Hall phenomenon is true and that it indeed relies upon the existence of collective particle momentum and the motion of the associated generalized flux quanta.

We thank Grisha Volovik for a discussion of the contents of this work. U. R. Fischer acknowledges financial support by the DFG (FI 690/1-1).

REFERENCES

[1] S. Das Sarma, A. Pinczuk (Eds.): *Perspectives in Quantum Hall Effects* (John Wiley & Sons, New York, 1997)

[2] We remark that in the literature, *e.g.* in J. D. Jackson: *Classical Electrodynamics*, Third Edition (John Wiley & Sons, 1998) what is called the "electromotive force" is actually an electromotive potential, *i.e.* a line integral of the electromotive field $\mathcal{E}$ along some path. We prefer, for obvious reasons, to call our $\mathcal{E}$ an electromotive force field. Correspondingly, $v \times \mathcal{B}$ is called the magnetomotive force field.

[3] B. L. Johnson: *Inertial forces and the Hall effect*, Am. J. Phys. **68**, 649-653 (2000)

[4] J. E. Zimmerman, J. E. Mercereau: *Compton Wavelength of Superconducting Electrons*, Phys. Rev. Lett. **14**, 887-888 (1965)

[5] J. Tate, S. B. Felch, B. Cabrera: *Determination of the Cooper-pair mass in niobium*, Phys. Rev. B **42**, 7885-7893 (1990)

[6] M. Liu: *Rotating Superconductors and the Frame-Independent London Equation*, Phys. Rev. Lett. **81**, 3223-3226 (1998)

[7] E. J. Post: *Sagnac Effect*, Rev. Mod. Phys. **39**, 475-493 (1967)

[8] G. E. Stedman: *Ring-laser tests of fundamental physics and geophysics*, Rep. Prog. Phys. **60**, 615-688 (1997)

[9] D. C. Tsui: *The Strange Physics of Two-Dimensional Electrons in a Strong Magnetic Field*, Physica B **164**, 59-66 (1990)

[10] U. R. Fischer: *Motion of Quantized Vortices as Elementary Objects*, Ann. Phys. (N.Y.) **278**, 62-85 (1999)

[11] The fact that one has $l_\Omega \gg l_\phi$, for any realizable (*i.e.* nondestructive) rotation speed, was disregarded in [3], where phase coherence was assumed to hold over the entire sample size, rather than over the distance l_ϕ.

[12] Note that using this argument for the fractional quantum Hall effect in a rotating sample, the *number* of particles, not their charge, flowing into (or out of) the region enclosed by Γ, were required to be nonintegral if exactly one quantum of flux is turned on.

[13] R. B. Laughlin: *Quantized Hall conductivity in two dimensions*, Phys. Rev. B **23**, 5632-33 (1981)

[14] A. Hartland *et al.*: *Direct Comparison of the Quantized Hall Resistance in Gallium Arsenide and Silicon*, Phys. Rev. Lett. **66**, 969-973 (1991)

[15] M. Krusius: *The Vortices of Superfluid* ^{3}*He*, J. Low Temp. Phys. **91**, 233-273 (1993)

[16] G. E. Volovik: *An analog of the quantum Hall effect in a superfluid* ^{3}*He-A Film*, JETP **67**, 1804-1811 (1988)

[17] F. Wilczek: *Fractional Statistics and Anyon Superconductivity*, World Scientific, 1990